Spin structure on moduli space of sheaves on Calabi-Yau threefold

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Abstract

Kontsevich and Soibelman defined a notion of orientation data on Calabi-Yau category. It can be viewed as a consistent choice of spin structure on moduli space of objects in the given CY category. The orientation data plays an important role in Donaldson-Thomas theory. Let X be a compact CY 3-fold satisfying appropriate topological condition. We prove the existence and uniqueness of orientation data on the derived category of coherent sheaves $\mathrm{D}^b(X)$.

1 Introduction

The goal of this paper is to construct $\mathscr{C}^\infty$ orientation data (See Definition 7.2) on moduli spaces of coherent sheaves on Calabi-Yau threefolds. Orientation data is introduced by Kontsevich and Soibelman in [12]. Roughly speaking, an orientation data on a Calabi-Yau category $\mathcal{C}$ (defined in section 3.3 of [12]) is a choice of square root of the determinant line bundle on the moduli space of objects, satisfying an additional compatibility condition. Depending on the structure we put on the determinant line bundle, one can consider different types of square roots. In the original definition of Kontsevich and Soibelman, they consider the determine bundle as a (ind-)constructible super line bundle over the moduli space of all objects in $\mathcal{C}$. We will not study in such a generality. Let $\mathcal{C}$ be $\mathrm{D}^b(coh(X))$, the derived category of coherent sheaves on a smooth projective CY 3-fold X. Moreover, we will restrict to moduli stack of coherent sheaves, denoted by $\mathcal{M}_X$ ($\mathcal{M}$ for short), instead of moduli of arbitrary complexes. One can stratify $\mathcal{M}$ such that over each strata there is a graded vector bundle with fiber $\mathrm{Ext}^\bullet_X(E,E)$ at point $E \in \mathcal{M}$. By abusing notations, we denote this (ind-)constructible graded vector bundle by $\mathrm{Ext}^\bullet_X(E,E)$. In particular, this means rank of $\mathrm{Ext}^i_X(E,E)$ is constant for all i over each strata. We define the (ind-)constructible determinant line bundle $\mathscr{L}$ to be $sdet(\mathrm{Ext}^\bullet_X(E,E))$ where

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sdet is the super determinant. Kontsevich-Soibelman's orientation data is a choice of (ind-)constructible square root of $\mathscr{L}$ that is compatible under Hall multiplication. The orientation data is a critical ingredient in the definition of motivic Donaldson-Thomas invariant (See section 6 [12]).

The (ind-)constructible structure is the weakest structure that one can put on determinant line bundle. However, it is sufficient for the purpose of [12] since motivic DT invariants are defined by first stratify the moduli stack and then add the different strata with appropriate motivic weights. By the work of Quillen and Knudsen-Mumford, the determinant line bundle can be equipped with a holomorphic or algebraic structure, without stratifying the moduli space. We will review these constructions in Section 2. Given the holomorphic determinant line bundle $\mathscr{L}$ over the underlying analytic stack associated to $\mathcal{M}$, a $\mathscr{C}^\infty$ *orientation data* is a choice of square root of $\mathscr{L}$ that is compatible under Hall multiplication (See Definition 7.2). From the point of view of derived geometry, the determinant line bundle $\mathscr{L}$ plays the role of canonical bundle of moduli space. We call a square root of $\mathscr{L}$ a *spin* structure on $\mathcal{M}$.

The existence and classification of $\mathscr{C}^\infty$ orientation data on moduli space is a purely topological question, essentially determined by the homotopy type of X. In this paper, we consider a special class of CY 3-folds called *admissible* CY 3-folds (Definition 6.8). The main theorem of the paper is:

Theorem 1.1. *(Theorem 7.6) If X is an admissible CY 3-fold, then there exists a $\mathscr{C}^\infty$ orientation data on $\mathrm{D}^b(X)$.*

Admissible condition is a condition on torsion part of homology of X. Any simply connected torsion free CY threefold is admissible (Theorem 6.10).

The proof of Theorem 7.6 is a combination of gauge theory and surgery theory. The same technique has been used widely in the study of four manifold. A very good reference is chapter 5 of [6]. In the first step, we reduce the question about moduli space of coherent sheaves to moduli of vector bundles. This is essentially consequence of a theorem of Joyce and Song (Theorem 7.3). Using gauge theory we realize the moduli space of holomorphic vector bundles as analytic subspace of a complex Banach manifold modulo automorphisms. A simple but important observation is that the determinant line bundle $\mathscr{L}$ extends to the ambient Banach manifold. Instead of constructing square root of $\mathscr{L}$ on the analytic subspace, we do it on the ambient Banach manifold. This is equivalent with proving $c_1(\mathscr{L})$ is divisible by two. Over rational number, it follows from Grothendieck-Riemann-Roch theorem and Atiyah-Singer index theorem (Theorem 3.1, Theorem 5.1). The torsion part of $c_1(\mathscr{L})$ is not captured by GRR or AS. We get around it by proving the even torsion cannot occur if X is admissible. The odd torsion never matters anway. To be more specific, we consider those CY 3-folds that can be uniformized to connected sum of $S^3 \times S^3$ by a sequence of conifold transitions and study how the torsion part of the second cohomology group of the Banach manifold mentioned above changes under the conifold transition.

Moduli space of sheaves on CY 3-fold is locally an intersection of two holomorphic Lagrangian subvarieties inside holomorphic symplectic manifold. Donaldson-

Thomas invariant is the (weighted) sum of Lagrangian intersection numbers. There are two generalizations of DT invariants. The first one is to replace the DT invariants, which are numbers, by elements in appropriate Grothendieck ring of stacks. Such generalization, due to Joyce-Song and Kontsevich-Soibelman, is called the motivic DT invariants ([11][12]). The second way is to replace it by a perverse sheaf on $\mathcal{M}$. Such perverse sheaf is known to exist locally by the work of Joyce and Song [11]. However, the gluing problem of these locally defined perverse sheaves is nontrivial. We refer to [3] and [13] for some recent progress. In both of these two generalized DT theories, the orientation data, i.e. spin structure on $\mathcal{M}$ plays an essential role.

This paper is organized as follows. In section 2, we recall two definitions of determinant line bundle. In section 3 and 5, we prove the rational $c_1(\mathscr{L})$ is divisible by two. In section 4, we recall some results in topology of loop spaces. In section 6, we define admissible CY 3-fold study the torsion in cohomology of space of principal bundles. The main theorem 7.6 is proved in Section 7. In the last section, we explain a link between orientation data and volume form on Lagrangian distribution which is used to categorify DT invariants. This section is independent from other sections. The readers might consider reading it first to get some motivations.

Acknowledgement The existence of orientation data is proved in the case of representations of quiver with potential by Davison [5]. Theorem 3.1 is known to Kontsevich. Thanks to Dominic Joyce for pointing out that. I would like to thank Conan Leung, Dominic Joyce for some very helpful discussion. Thanks to Prof. Simon Donaldson for his comments about η-invariants. Thanks to Young-Hoon Kiem for pointing out a mistake in the earlier version. Part of the work was done during my visit to Oxford. I am grateful to Balazs Szendroi for his hospitality.

2 Determinant line bundle

We recall two definitions of determinant line bundle, one in algebraic geometry and one in differential geometry. Both definitions will be used later.

Let $\mathcal{M}_X$ ($\mathcal{M}$ for short) be the moduli space of sheaves on X. This is an Artin stack locally of finite type. One can write $\mathcal{M}$ as disjoint union indexed by classes in topological K-theory:

$$\mathcal{M} = \bigsqcup_{\beta \in K_0(X)} \mathcal{M}_\beta.$$

2.1 Algebraic definition

Let $\mathcal{E}$ be the *universal sheaf* over $\mathcal{M} \times X$. Denote the projection $\mathcal{M} \times X \to \mathcal{M}$ by π.

Definition 2.1. The (algebraic) determinant line bundle $\mathscr{L}$ over $\mathcal{M}$ is defined to be

$$\mathrm{sdet}(\pi_* \mathbf{R}\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})[1])$$

where $\mathbf{R}\underline{\mathrm{Hom}}$ is the sheaf derived endomorphism.

Because $\mathcal{E}$ is flat over $\mathcal{M}$ and X is smooth, $\mathbf{R}\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})$ is a perfect complex. The determinant is well defined.

Let E be a sheaf over X. Its equivalence class $[E]$ represents a point in the moduli stack $\mathcal{M}$. The *tangent complex* of $\mathcal{M}$ at $[E]$ is defined to be the graded vector space

$$T_{[E]}\mathcal{M} = \bigoplus_i \mathrm{Ext}^i_X(E, E).$$

It has an obvious graded Lie algebra structure by anti-commuting the associative product. The degree zero piece is the subalgebra of endomorphisms whose corresponding group is $Aut(E)$.

The fiber of the determinant line bundle $\mathscr{L}$ at $[E]$ is

$$(\bigwedge^{top} \mathrm{Ext}^{even}(E, E))^{-1} \otimes \bigwedge^{top} \mathrm{Ext}^{odd}(E, E).$$

It inherits an action of $Aut(E)$.

A sheaf E is called *simple* if $\mathrm{Ext}^0(E, E) \cong \mathbb{C}$. We denote the moduli space of simple sheaves by $\mathcal{M}^{si}$. It is a $\mathbb{C}^*$ gerb over a scheme (Deligne-Mumford stack) locally of finite type. By restricting to sheaves with fixed determinant we can remove this $\mathbb{C}^*$ gerb consistently and obtain a moduli scheme (DM stack). The determinant line bundle descends.

2.2 Analytic definition

We give a second definition of $\mathscr{L}$ for moduli space of holomorphic vector bundles. Our main reference is [7] and [15]

Let E be a complex vector bundle over X. We identify it with the frame bundle of a principal G^c bundle for $G^c = GL(n, \mathbb{C})$ where n is the rank of E. Denote $\mathscr{A}$ for the infinite dimensional linear manifold $\mathscr{A}^{0,1}(ad\ E)$ of $(0,1)$ forms with value in adjoint bundle. An infinite dimensional gauge group $\mathscr{G}^c := \mathscr{C}^\infty(X, G^c)$ acts on $\mathscr{A}$. The orbit space $\mathscr{A}/\mathscr{G}^c$ parameterizes equivalence classes of $(0,1)$-connections. Typically, we complete $\mathscr{A}$ to a complex Banach space under appropriate Sobolev norm. However, the results in this paper are insensitive to particular choice of Sobolev norm. So we will ignore the completion.

A $(0,1)$ connection $\nabla : \mathscr{A}^{0,0}(E) \to \mathscr{A}^{0,1}(E)$ is called *integrable* if its curvature $F_\nabla = \nabla^2$ vanishes. Strictly speaking, $\mathscr{A}$ is identified with the underlying affine space of $\mathscr{A}^{0,1}(ad\ E)$. Given two $(0,1)$-connections A and A', their difference $A - A'$ lies in $\mathscr{A}^{0,1}(ad\ E)$.

If we fix an integrable reference connection ∇, a connection $\nabla_A := \nabla + A$ for $A \in \mathscr{A}^{0,1}(ad\ E)$ is integrable if and only if it satisfies the *Maurer-Cartan* equation

$$d_\nabla A + A \wedge A = 0$$

where $d_\nabla := [\nabla, -]$. The space of integrable connections, denoted by $\mathscr{A}^{(1,1)}$, is an analytic subspace of $\mathscr{A}$. Because integrable $(0,1)$ connections are in one to one correspondence with holomorphic structures on E, the Banach analytic stack $\mathscr{A}^{(1,1)}/\mathscr{G}^c$ parameterizes equivalence classes of holomorphic vector bundles with underlying complex vector bundle being E.

If we pick a hermitian metric on E, then the gauge group G^c is reduced to the unitary group $G = U(n)$. Similar to the fixing determinant trick in algebraic geometry, we can consider the gauge group to be special unitary group. By abusing the notations, we use E to denote the underlying principal $SU(n)$ bundle.

Given a connection A, there is a first order elliptic operator D_A acting on the Dolbeault complex $L := \mathscr{A}^{0,\bullet}(adE)$, defined as

$$D_A = \nabla_A + \nabla_A^* : L^{even} \to L^{odd}$$

where ∇_A^* is the adjoint with respect to the L^2 metric induced by the hermitian metric on E. The adjoint D_A^* maps L^{odd} to L^{even}.

When ∇_A is integrable, there are isomorphisms

$$Ker\ D_A \cong \mathrm{Ext}^{even}(E_{\nabla_A}, E_{\nabla_A}), Ker\ D_A^* \cong \mathrm{Ext}^{odd}(E_{\nabla_A}, E_{\nabla_A})$$

where E_{∇_A} is the corresponding holomorphic vector bundle. When ∇_A is non-integrable, D_A is invertible.

Definition 2.2. For the family of elliptic operators D_A with $A \in \mathscr{A}$, we define the fiber of its determinant line bundle $\mathscr{L}$ at A to be

$$(\bigwedge^{top} Ker\ D_A)^{-1} \otimes \bigwedge^{top} Ker\ D_A^*.$$

One can turn $\mathscr{L}$ into a holomorphic line bundle over $\mathscr{A}$. Define the $\bar{\partial}$-Laplacian Δ_A to be the second order elliptic operator $D_A D_A^*$. For $A \in \mathscr{A}$ and a positive real number l that is not in the spectrum of Δ_A, there exists an open neighborhood U_A of A such that the direct sum of eigenspaces of Δ_A

$$H^+_{<l} = \bigoplus_{\lambda<l} H^+_\lambda$$

and

$$H^-_{<l} = \bigoplus_{\lambda<l} H^-_\lambda$$

form holomorphic vector bundles over U_A. The determinant line bundle $\mathscr{L}$ is defined locally to be

$$(\bigwedge^{top} H^+_{<l})^{-1} \otimes \bigwedge^{top} H^-_{<l}.$$

Because D_A is an isomorphism from H^+_λ to H^-_λ for λ positive, the above definition is independent with choice of l.

The elliptic operator D_A restricts to a linear transform from $H^+_{<l}$ to $H^-_{<l}$. Its determinant $det(D_A)$ defines a section of $\mathscr{L}$ over U_A that vanishes exactly when $Ker(D_A)$ is nonzero. On the overlap of two charts U_A and $U_{A'}$, there are two numbers $0 < l < l'$ that are not in the spectrum of Laplacians such that $\mathscr{L}$ are super determinants defined above. The section $det(D_A)$ of $(\det H^+_{(l,l')})^{-1} \otimes \det H^-_{(l,l')}$ is invertible. By multiplying such section, the determinant line bundle glues on the overlap.

It is clear that the above construction is compatible with action of $\mathscr{G}^c$, i.e. $\mathscr{L}$ is a line bundle over the stack $\mathscr{A}/\mathscr{G}^c$.

When both algebraic and analytic definitions of the determine line bundle apply, they coincide.

Remark 2.3.

1. The algebraic definition applies to any perfect complexes, in particular coherent sheaves, over X. The analytic definition only applies for vector bundles.

2. The analytic definition applies for family (not necessarily bounded) of elliptic operators on space of sections of vector bundle. When the elliptic operator is Δ_A for an integrable connection A, it coincides with the algebraic definition fiberwise. However, the determinant still makes sense when F_A is nonzero while the algebraic definition stops to work.

The key observation of the paper is that the analytic definition of determinant bundle is a better definition to discuss Kontsevich-Soibelman's orientation data. This is because by working with non-integrable connections most technical difficulties coming from singularities of moduli space are gone.

3 Rational $c_1(\mathscr{L})$ I

Let $\mathcal{M}$ and $\mathscr{L}$ be defined as above. If $\mathcal{M}$ is a smooth manifold then first Chern class $c_1(\mathscr{L})$ lies in $H^2(\mathcal{M}, \mathbb{Z})$. $\mathscr{L}$ has a topological square root if and only if $c_1(\mathscr{L})$ is divisible by two.

Over rational, $c_1(\mathscr{L})$ can be computed by Grothendieck-Riemann-Roch theorem in algebraic context or Atiyah-Singer index theorem in analytic context. It suffices to check the case when $\mathcal{M}$ is a compact Riemann surface. In this section, we compute the rational c_1 using GRR theorem, under the assumption that $\mathcal{M}$ is a smooth proper curve. Proof of the general case, which requires Atiyah-Singer index theorem, will be given in section 5.

Let X be a simply connected CY 3-fold and C be a compact Riemman surface. Let $\mathcal{E}$ be the universal sheaf over $X \times C$, π be the projection to C and p be the projection to X.

Theorem 3.1. *Modulo torsion, the first Chern class of* $\pi_* \mathbf{R}\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})$ *is divisible by* 2.

Proof. For simplicity we denote F for $\mathbf{R}\underline{\mathrm{Hom}}(\mathcal{E},\mathcal{E})$. GRR theorem says

$$ch(\pi_! F)\cdot td_C = \pi_*(ch(F)\cdot td_{X\times C}).$$

By adjunction, we can rewrite it as

$$ch(\pi_! F) = \pi_*(ch(F)\cdot p^* td_X).$$

The Todd class of X is

$$1+\frac{c_2(X)}{12}.$$

The Chern character of $\mathcal{E}$ is

$$ch(\mathcal{E}) = r + c_1 + \frac{c_1^2-2c_2}{2} + \frac{c_1^3-3c_1c_2+3c_3}{6} + \frac{c_1^4-4c_1^2c_2+2c_2^2+4c_1c_3-4c_4}{24}.$$

The Chern character of F is

$$ch(F) = r^2+((r-1)c_1^2-2rc_2)+\frac{(r-1)c_1^4-4rc_1^2c_2+2(r+6)c_2^2+4(r-3)c_1c_3-4rc_4}{12}.$$

Apply GRR, we obtain

$$c_1(\pi_! F) = [\frac{(r-1)(c_1^4+c_1^2c_2(X))}{12} - \frac{rc_2c_2(X)}{6} - \frac{rc_1^2c_2}{3} + \frac{(r+6)c_2^2}{6} + \frac{(r-3)c_1c_3}{3} - \frac{rc_4}{3}]\cdot[X].$$

Separate terms depending on rank and terms independent of rank.

$$\begin{aligned} c_1(\pi_! F) &= r[\frac{c_1^2c_2(X)+c_1^4}{12} - \frac{c_2c_2(X)}{6} + \frac{c_2^2}{6} - \frac{c_1^2c_2}{3} + \frac{c_1c_3}{3} - \frac{c_4}{3}]\cdot[X] - [\frac{c_1^2c_2(X)+c_1^4}{12} - c_2^2 + c_1c_3]\cdot[X] \\ &= 2rc_1(\pi_!\mathcal{E}) - [\frac{c_1^2c_2(X)+c_1^4}{12} - c_2^2 + c_1c_3]\cdot[X] \end{aligned} \tag{3.1}$$

The rank depending term is even since $c_1(\pi_!\mathcal{E})$ belongs to $H^2(C,\mathbb{Z})$. We need to show the rank independent term is even.

Lemma 3.2. *Let A be a class in $H^2(X,\mathbb{Z})$. Then*

$$2A^3 + A\cup c_2(X) \equiv 0 \ mod\ 12.$$

Proof. Let A be $c_1(D)$ for some divisor D on X. By GRR,

$$\chi(\mathcal{O}_X(D)) = \frac{A^3}{6} + \frac{A\cup c_2(X)}{12}.$$

Since this must be an integer, the lemma follows. □

By Kunneth formula, c_1 can be written as $p^*A+\pi^*B$ where $A\in H^2(X,\mathbb{Z})$ and $B\in H^2(C,\mathbb{Z})$. By previous lemma, the term

$$\frac{c_1^2c_2(X)+c_1^4}{12}\cdot[X] = \frac{2ABc_2(X)+4A^3B}{12}\cdot[X].$$

is even.

Suppose r is odd, there exists a divisor D such that $c_1(\mathcal{E}(D))$ is even. Therefore, we can assume c_1c_3 to be even. Finally, c_2^2 is even again by Kunneth formula.

If $\mathcal{E}$ is a line bundle, the theorem holds trivially. Let $\mathcal{E}$ be a sheaf of rank $r = 2k > 0$, we consider a short exact sequence

$$0 \longrightarrow L \longrightarrow \mathcal{E} \longrightarrow Q \longrightarrow 0$$

where L is a line bundle and Q is a sheaf of rank $2k-1$.

$$ch(\mathcal{E}^\vee \otimes \mathcal{E}) = ch(L^\vee \otimes L) + ch(Q^\vee \otimes Q) + ch(L^\vee \otimes Q) + ch(L \otimes Q^\vee).$$

The sum of the last two terms is even because the odd terms get canceled and the even terms get doubled. Then the even rank case is proved.

□

4 Some results in topology and gauge theory

We recall some basics on gauge theory. Our main reference is Chapter 5 of [6]. Let X be a compact complex manifold and G be a compact Lie group. Unless we specify, G will be taken to be $SU(n)$.

Let P be a principal G bundle over X. Let $\mathscr{A} = \mathscr{A}_{X,P}$ be the space of connections on P and $\mathscr{G}$ be the gauge group. The main theme of this section is the topology of the orbit space $\mathscr{B} = \mathscr{A}/\mathscr{G}$. Sometime, when there is doubt about which manifold is involved, we write $\mathscr{B}_X$ instead.

Because the action of $\mathscr{G}$ is not free, it is much easier to work with *framed* connections. If x_0 is a base point on X, a framed connection is a pair (A, ϕ) where A is a connection and ϕ is an isomorphism of G-spaces $\phi : G \to P_{x_0}$. The gauge group $\mathscr{G}$ acts naturally on space of framed connections and we write $\tilde{\mathscr{B}}$ for the space of equivalence classes

$$\tilde{\mathscr{B}} = (\mathscr{A} \times \mathrm{Hom}(G, P_{x_0}))/\mathscr{G}.$$

There is a natural action of the finite dimensional gauge group G on $\tilde{\mathscr{B}}$ such that the quotient stack is $\mathscr{B}$. One way to think of this quotient is to regard a framing ϕ as fixed and define $\mathscr{G}_0 \subset \mathscr{G}$ to be its stabilizer, that is

$$\mathscr{G}_0 = \{g \in \mathscr{G} | g(x_0) = 1\}.$$

Then $\tilde{\mathscr{B}}$ may be described as $\mathscr{A}/\mathscr{G}_0$ and the projection $\tilde{\mathscr{B}} \to \mathscr{B}$ is simply the quotient map for the remainder of the gauge group,

$$\mathscr{G}/\mathscr{G}_0 \cong G.$$

We recall a standard theorem in algebraic topology.

Theorem 4.1. *There is a weak homotopy equivalence*

$$\tilde{\mathscr{B}}_{X,P} \simeq Map^*(X, BG)_P$$

where Map^* denotes base-point-preserving maps and $Map^*(X, BG)_P$ denotes the homotopy class corresponding to the bundle $P \to X$.

Proof. See proposition 5.1.4 of [6]. □

Proposition 4.2. *When G equals $SU(n)$ for $n \gg 0$, $H^2(\tilde{\mathscr{B}}_{S^6}, \mathbb{Z})$ is torsion free.*

Proof. Because a principal G bundle over S^6 is determined by its transition function over the equator, we have a homotopy equivalence

$$\tilde{\mathscr{B}}_{S^6} \simeq Map^*(S^5, SU(n)) = \Omega^5 SU(n).$$

Lemma 4.3. *Assume $k < 2n$. There is an isomorphism*

$$H^N(\Omega^k SU(n)) \cong H^N(\Omega^k SU(n+1))$$

for $N < 2n - k$.

Proof. We first compute the cohomology of iterated loop spaces of odd spheres since they are the building blocks of loop spaces of $SU(n)$. When $k < 2n+1$, we claim

$$H^j(\Omega^k S^{2n+1}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & j = 0, 2n+1-k \\ 0 & 0 < j < 2n+1-k \end{cases} \tag{4.1}$$

This can be proved by induction on k. When $k = 0$, this is clearly right. Assume k is even. Consider the fibration

$$\Omega^k S^{2n+1} \to P\Omega^{k-1} S^{2n+1} \to \Omega^{k-1} S^{2n+1}.$$

The condition $k < 2n+1$ guarantees the base to be connected and simply connected. By the induction assumption, the bottom row of E^2 page of the Serre spectral sequence looks like

$$\mathbb{Z} \qquad \ldots 0 \ldots \qquad \mathbb{Z} \qquad \ldots 0 \ldots$$

where the second copy of $\mathbb{Z}$ appear in degree $2n+2-k$. Since the path space is contractible, the claim follows. The other half of the claim can be check similarly.

Consider the spectral sequence for the fibration

$$\Omega^k SU(n) \to \Omega^k SU(n+1) \to \Omega^k S^{2n+1}.$$

The j-th row of E^2 page is zero when $j = 2, 3, \ldots, 2n-k$. The convergence of spectral sequence gives $H^N(\Omega^k SU(n+1)) \cong H^N(\Omega^k SU(n))$ when $N = 0, 1, \ldots, 2n-1-k$.

Remark 4.4. The integral cohomology of $\Omega^k S^{2n+1}$ is in general not torsion free. However, for big n the lower degree component will be torsion free.

$\square$

Recall the *stable* special unitary group SU is defined to be the direct limit

$$SU = \varinjlim SU(n).$$

In general, the cohomology functor doesn't commute with direct limit. The discrepancy is measured by the derived functor $\varinjlim^1$. However, by Lemma 4.3 $\varinjlim^1$ vanishes. Therefore, for fixed N and k

$$H^N(\Omega^k SU) \cong H^N(\Omega^k SU(n))$$

when $n \gg 0$.

The Bott periodicity theorem says there is a homotopy equivalence

$$\Omega SU \simeq BSU.$$

We are interested in the first and the second cohomology group of $\Omega^5 SU(n)$, i.e. $N = 1, 2$. By Bott periodicity, $H^N(\Omega^5 SU(n), \mathbb{Z}) \cong H^N(\Omega^5 SU, \mathbb{Z}) \cong H^N(\Omega SU, \mathbb{Z})$ when $n \gg 0$. By Bott's theorem [4], ΩSU is torsion free and generated by universal Chern classes. As a consequence, $H^1(\tilde{\mathscr{B}}_{S^6}, \mathbb{Z}) = 0$ and $H^2(\tilde{\mathscr{B}}_{S^6}, \mathbb{Z}) = \mathbb{Z}$ for $G = SU(n)$ with $n \gg 0$. $\square$

Because $\tilde{\mathscr{B}}_{S^2} \simeq Map^*(S^2, BSU(n)) \simeq \Omega SU(n)$, its integral cohomology is torsion free and generated by the universal Chern classes $c_1, \ldots, c_n$. This is a special case of Proposition 2.20 of [1].

Similarly, $\tilde{\mathscr{B}}_{S^3}$ is homotopic to $\Omega^2 SU(n)$. By Lemma 4.3, when $n \gg 0$ $H^i(\Omega^2 SU(n)) = H^i(SU) = 0$ for $i = 1, 2$ and $H^3(\Omega^2 SU(n)) = H^3(SU) = \pi_3(S^3) = \mathbb{Z}$.

5 Rational $c_1(\mathscr{L})$ II

In this section, we prove theorem 3.1 for $\mathcal{M}$ not necessarily smooth scheme.

Let X be a simply connected CY 3-fold. As we see in section 2, there is a family of elliptic operators D_A over the space of framed connections $\tilde{\mathscr{B}}_X$. The family has determinant line bundle $\mathscr{L}$.

Denote the index bundle of the elliptic complex

$$L^{even} \underset{D_A^*}{\overset{D_A}{\rightleftarrows}} L^{odd}$$

by $ind(D, E)$. We are interested in the case of index bundle of the universal principal bundle $\mathcal{P}$ over $\tilde{\mathscr{B}}_X \times X$.

The family version of Atiyah-Singer index theorem says

$$ch(ind(D, ad(\mathcal{P}))) = (ch(ad(\mathcal{P}))\hat{A}(X))/[X].$$

Here $\hat{A}(X)$ is equal to $1 + \frac{p_1(X)}{24}$ with $p_1(X)$ being the first Pontryagin class of X. The notation $/[X]$ means the slant product.

The orbit space $\mathscr{B}$ is a smooth stack $\tilde{\mathscr{B}}/G$. The determinant line bundle $\mathscr{L}$ over $\tilde{\mathscr{B}}$ is G-equivariant. We need to calculate the equivariant c_1 of $\mathscr{L}$. As a cohomology class, it lies in the equivariant cohomology group $H^2_G(\tilde{\mathscr{B}}, \mathbb{Z})$. By definition, it is equal to $H^2(\tilde{\mathscr{B}} \times_G EG, \mathbb{Z})$. Because both $\tilde{\mathscr{B}}$ and G are smooth manifolds, the homotopy fiber space $\tilde{\mathscr{B}} \times_G EG$ has a model as an infinite dimensional manifold.

Because the index bundle $ind(D, ad(\mathcal{P}))$ is G-equivariant, it has a lifting to a bundle over $\tilde{\mathscr{B}} \times_G EG$. We want to compute

$$c_1(\mathscr{L}) = c_1(ind(D, ad(\mathcal{P}))) = (ch(ad(\mathcal{P}))\hat{A}(X))/[X] - 1.$$

Apply Atiyah-Singer index theorem , the calculation reduces to the case of $\mathcal{P}$ over $X \times C$ where C is a compact Riemann surface. The same computation in section 3 gives:

Theorem 5.1. *Modulo torsion, $c_1(\mathscr{L})$ is divisible by two.*

6 Conifold transition

Suppose X^+ and X^- are two (real) six manifolds relating by a conifold transition. In this section, we will study the relations between integral cohomologies of $\tilde{\mathscr{B}}_{X^+}$ and $\tilde{\mathscr{B}}_{X^-}$.

First we recall the definition of *conifold transition*. Let X^- be a six manifold, compact and orientable. Given an embedded 3-sphere in X^-, if X^- is CY then the tubular neighborhood of this S^3 can be identified with a manifold U that is the (open) D^3 bundle in the cotangent bundle of S^3. Its closure $\bar{U}$ is a six manifold with boundary being a 2-sphere bundle over S^3. A simple calculation shows this sphere bundle is trivial.

On the other hand, we consider the total space of a rank two complex vector bundle over S^2 that is the direct sum of two complex line bundles of Chern class -1. Denote the (open) D^4 bundle inside this vector bundle by V. Its closure $\bar{V}$ is a six manifold with boundary being a 3-sphere bundle over S^2. Again, one can show this is a trivial S^3 bundle. We glue $X^- \setminus U$ with $\bar{V}$ along the boundary by the isomorphism $\partial \bar{U} \cong \partial \bar{V} \cong S^2 \times S^3$ and denote the resultant six manifold by X^+. The above surgery is called a *positive* conifold transition. A *negative* conifold transition is defined by starting with an embedded S^2 and reverse the above engineering.

With enough care, conifold transition can be operated in the categories of complex, symplectic, Kahler or algebraic manifolds. Furthermore, it preserves the CY condition. However, in this section we will not refer to any additional structure other than smooth manifold.

We give a few examples of conifold transition.

Example 6.1. Let X^- be $S^3 \times S^3$. The positive conifold transition at one of the S^3 in the product produces $X^+ = S^6$.

Example 6.2. Let X^- be connected sum of k copies of $S^3 \times S^3$. Pick S^3 to be one copy of $S^3 \times S^3$ and do positive conifold transition. The resultant six manifold is the connected sum of $k-1$ copies of $S^3 \times S^3$.

Now we give one example of conifold transition between CY 3-folds. It is borrowed from [16].

Example 6.3. Let $\mathbb{C}[x_0, \ldots, x_4]$ be the homogeneous coordinate ring of $\mathbb{P}^4$. Consider a degree five hypersurface defined by $x_3 g(x_0, \ldots, x_4) + x_4 h(x_0, \ldots, x_4) = 0$. It contains a plane defined by $x_3 = x_4 = 0$. If we choose g and h general enough, the singular set of the hypersurface will be the intersection of two generic quartic curves in this plane, i.e. 16 isolated points. It is easy to check they are nodal singularities. Let's denote this singular hypersurface by $\bar{Y}$.

One can find a Weil divisor passing through all these 16 nodes. The blow-up along this Weil divisor gives a small resolution of $\bar{Y}$, denoted by Y. The pre-images of the 16 nodes in Y are 16 smooth rational curves. On the other hand, we can smooth out $\bar{Y}$ in the linear system of quintics and obtain a smooth quintic threefold $\widetilde{Y}$. There will be 16 Lagrangian vanishing 3-spheres in $\widetilde{Y}$.

Hodge numbers of Y and $\widetilde{Y}$ are

$$h^{1,1}(Y) = 2, h^{2,1}(Y) = 86;$$

$$h^{1,1}(\widetilde{Y}) = 1, h^{2,1}(\widetilde{Y}) = 101.$$

This implies immediately that the 16 exceptional rational curves on Y are not homologically independent and the 16 Lagrangian spheres on $\tilde{Y}$ are not homologically independent either.

To go from Y to $\widetilde{Y}$, we do one negative conifold transition at one of the 16 rational curves. The Lagrangian vanishing sphere produced in this step is homologically trivial, i.e. the rank of H^4 drops by one while the rank of H^3 stays the same. Now all the 15 exceptional S^2 rest are homologically trivial. We do negative transition for 15 times. Every time, the H^4 stays the same while rank of H^3 increases by one since the vanishing sphere will be homologically nontrivial.

Every time we do a conifold transition, there are two cofibrations.

$$S^2 \to X^+ \to X;$$

$$S^3 \to X^- \to X.$$

Because the mapping functor turns cofibration to fibration, we obtain two fibrations

$$\tilde{\mathscr{B}}_X \to \tilde{\mathscr{B}}_{X^+} \to \tilde{\mathscr{B}}_{S^2};$$
$$\tilde{\mathscr{B}}_X \to \tilde{\mathscr{B}}_{X^-} \to \tilde{\mathscr{B}}_{S^3}.$$

Theorem 6.4. *Let X^+ be a six manifold, compact, simply connected and orientable such that $\tilde{\mathscr{B}}_{X^+}$ has no torsion in its first and second integral cohomologies. Denote X^- for the six manifold that is a negative conifold transition of X^+ at an embedded S^2. Then $\tilde{\mathscr{B}}_{X^-}$ has no torsion in its first and second integral cohomologies.*

Proof. Apply the results of section 4 and Serre spectral sequence to

$$\tilde{\mathscr{B}}_X \to \tilde{\mathscr{B}}_{X^+} \to \tilde{\mathscr{B}}_{S^2}.$$

The E^2 page looks like

$$\begin{tikzcd}
\star \arrow[drr] & 0 & \star & 0 \\
\star \arrow[drr, "\phi"] & 0 & \star & 0 \\
\mathbb{Z} & 0 & \mathbb{Z} & 0
\end{tikzcd}$$

Because the first and second cohomologies of $\tilde{\mathscr{B}}_{X^+}$ are torsion free and the spectral sequence degenerates at the E^3 page, the first and second cohomologies of $\tilde{\mathscr{B}}_X$ have to be torsion free as well.

The E^3 page of the spectral sequence of

$$\tilde{\mathscr{B}}_X \to \tilde{\mathscr{B}}_{X^-} \to \tilde{\mathscr{B}}_{S^3}.$$

looks like

$$\begin{tikzcd}
\star \arrow[ddrrr] & 0 & 0 & \star & 0 \\
\star & 0 & 0 & \star & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} & 0
\end{tikzcd}$$

Because the first and second cohomologies of the fiber are torsion free, so is $\tilde{\mathscr{B}}_{X^-}$. □

When $\tilde{\mathscr{B}}_{X^+}$ is simply connected $\tilde{\mathscr{B}}_{X^-}$ doesn't have to be so. The rank of H^1 might go up by one under negative conifold transition.

Theorem 6.5. *Let X^- be a six manifold, compact, simply connected and orientable such that $\tilde{\mathscr{B}}_{X^-}$ has no torsion in its first and second integral cohomologies. Denote X^+ for the six manifold that is a positive conifold transition of X^- at an embedded S^3. Then $\tilde{\mathscr{B}}_{X^+}$ has no torsion in its first cohomology and it has a p-torsion in its second cohomology if and only if the embedded S^3 is a p-th power of a primitive class in $H_3(X^-, \mathbb{Z})$.*

Proof. It follows from the argument in the previous theorem that the first and second cohomologies of $\tilde{\mathscr{B}}_X$ are torsion free.

The only map on the E^2 page of the spectral sequence associated to

$$\tilde{\mathscr{B}}_X \to \tilde{\mathscr{B}}_{X^+} \to \tilde{\mathscr{B}}_{S^2}.$$

that can create torsion is the map

$$\phi : H^1(\tilde{\mathscr{B}}_X) \to H^2(\tilde{\mathscr{B}}_{S^2}).$$

Lemma 6.6. *There is a surjection from $H_3(X^-, \mathbb{Z})$ to $H^1(\tilde{\mathscr{B}}_X, \mathbb{Z})$. So element of $H^1(\tilde{\mathscr{B}}_X, \mathbb{Z})$ can be represented by formal sum of three manifolds inside X^-. If we identify $H^2(\tilde{\mathscr{B}}_{S^2})$ with $\mathbb{Z}$, then evaluation of ϕ at a class of three manifold M is equal to the intersection number of M with the embedded S^3 inside X^-.*

Proof. By spectral sequence, $H^1(\tilde{\mathscr{B}}_X, \mathbb{Z})$ is equal to $H^1(\tilde{\mathscr{B}}_{X^-}, \mathbb{Z})$. We need to produce a map from $H_3(X^-, \mathbb{Z})$ to $H^1(\tilde{\mathscr{B}}_{X^-}, \mathbb{Z})$. Since the latter is torsion free, it suffices to work over the rational. There are two equivalent ways to define such a map, explained in [6]. Let's recall the construction. Let $\mathcal{P}$ be the universal principal bundle over $\tilde{\mathscr{B}}_{X^-} \times X^-$. The slant product $\mu_{c_2(\mathcal{P})}$ defines a map from $H_3(X^-)$ to $H^1(\tilde{\mathscr{B}}_{X^-})$. It can be realized as a map sending M to the differential of

$$A \mapsto \int_M Tr(CS(A))$$

where $CS(A) = dA \wedge A + \frac{2}{3} A \wedge A \wedge A$ is the Chern-Simons three form associate to A.

This is a map from $\tilde{\mathscr{B}}_{X^-}$ to S^1. Its differential defines an one form on $\tilde{\mathscr{B}}_{X^-}$. Because we assume X^- is simply connected, the existence of non-trivial 3-cycles is the only obstruction to vanishing of $H^1(\tilde{\mathscr{B}}_{X^-}, \mathbb{Z})$. Therefore, the μ map is surjective. A more general result can be found on page 181 of [6].

We compose the morphism $H_3(X^-) \xrightarrow{\mu} H^1(\tilde{\mathscr{B}}_{X^-})$ with $\phi : H^1(\tilde{\mathscr{B}}_X) \to H^2(\tilde{\mathscr{B}}_{S^2})$. Since $\tilde{\mathscr{B}}_{S^2}$ is simply connected, by Hurewicz theorem, $\pi_2(\tilde{\mathscr{B}}_{S^2}) = H_2(\tilde{\mathscr{B}}_{S^2})$. Therefore, ϕ is induced by the connecting morphism of the long exact sequence of homotopy groups

$$\pi_2(\tilde{\mathscr{B}}_{S^2}) \to \pi_1(\tilde{\mathscr{B}}_X).$$

Pick a point $a \in \tilde{\mathscr{B}}_{S^2}$, the fiber of $\tilde{\mathscr{B}}_{X^+}$ at a, denoted by F_a, consists of connections that restrict to a on S^2. Any 2-cycle on $\tilde{\mathscr{B}}_{S^2}$ lifts to a relative 2-cycle on $(\tilde{\mathscr{B}}_{X^+}, F_a)$. The connecting morphism is exactly the boundary map.

Let M be an embedded three manifold in X^-. Perturb it such that it intersects S^3 transversally. Under the positive conifold transition, M is replaced by $M^+ \subset X^+$, which is a three manifold with boundary. Each connected components of ∂M^+ is a copy of S^2 and the number of components (counted with sign that comes from orientation) is equal to the intersection number. Given a 1-cycle γ on F_a, the morphism $\phi \circ \mu : H_3(X^-) \to H^2(\tilde{\mathscr{B}}_{S^2})$ is the slant

product of $c_2(\mathcal{P})$ with the four manifold $\gamma \times M^+$ in $\tilde{\mathscr{B}}_{X^+} \times X^+$. It is equal to the integration of Chern-Simons three form on the boundary.

By spectral sequence computation, $H^2(\tilde{\mathscr{B}}_{S^2})$ is a subspace of $H^2(\tilde{\mathscr{B}}_{X^+})$. Its generator can be realized as the differential of a map from $H_1(\tilde{\mathscr{B}}_{X^+}) = H_1(\tilde{\mathscr{B}}_X)$ to S^1:

$$\gamma \mapsto \int_{S^2 \times \gamma} Tr(CS(A)).$$

Because the the restriction of connection on all the connected components of the boundary are the same, by Stokes theorem the slant product is equal to the number of components (with sign) times the above generator. So it is equal to the intersection number $[M] \cdot [S^3]$. □

If S^3 is p-th power of a primitive class in $H_3(X^-, \mathbb{Z})$, the intersection number $[M] \cdot [S^3]$ is divisible by p. Moreover, we can choose M such that the intersection number equals p. Then $Cok(\phi) = \mathbb{Z}/p\mathbb{Z}$. On the other hand, if the Cokernel is $\mathbb{Z}/p\mathbb{Z}$ then the S^3 must be p-th power of a primitive class.

□

Remark 6.7. The torsion part of $c_1(\mathscr{L})$ is the mathematical formulation of Witten's global anomaly. It can be computed using mod k index theorem of Freed and Melrose [8]. The topological index in the mod k situation involves the η invariant. The computation of η invariants is difficult since it relies on particular choice of metric and connection. In very rough sense, the η invariant that we need to compute are of the following form. Take γ to be a loop in $\tilde{\mathscr{B}}$ such that $\gamma^k \sim 0$. The determinant line bundle $\mathscr{L}$ has a natural connection ∇ defined by Quillen. The η-invariant is essentially determined by $hol_\nabla(\gamma)$. In the setup of mod k index theorem, we should consider γ as boundary of certain Riemann surface Σ inside $\tilde{\mathscr{B}}$. $hol_\nabla(\gamma)$ is not a topological quantity by itself. However, by adding a term that is integration of curvature form of $\mathscr{L}$ over Σ the sum becomes a topological quantity. If we choose the cylindrical metric on Σ, then we can make $hol_\nabla(\gamma)$ a topological quantity since ∇ is flat. We refer to [8] for details.

Let Y be a six manifold. By universal coefficient theorem, the torsion part of $H^2(\tilde{\mathscr{B}}_Y)$ is same as the torsion part of $H_1(\tilde{\mathscr{B}}_Y)$. The absence of torsion in H^2 forces the vanishing of η invariants.

Definition 6.8. Let Y be a simply connected compact CY 3-fold. We say Y is *admissible* if

- There exists a finite sequence of conifold transitions, positive or negative that transform Y to S^6;
- The positive conifold transition only involves S^3 that is odd (or zero) power of a primitive class.

Corollary 6.9. *If Y is an admissible CY 3-fold and $G = SU(n)$ for $n \gg 0$, then $H^1(\tilde{\mathscr{B}}_Y, \mathbb{Z})$ is torsion free and $H^2(\tilde{\mathscr{B}}_Y, \mathbb{Z})$ is free of even torsion.*

Proof. This follows from theorem 4.2, theorem 6.4 and theorem 6.5. $\square$

A manifold X is called torsion free if its integral homology is free of torsion.

Theorem 6.10. *Any simply connected torsion free CY 3-folds are admissible.*

Proof. Let X^+ be a simply-connected torsion free CY 3-fold. The basic idea is to do a sequence of negative conifold transitions to kill $H_2(X^+, \mathbb{Z})$. If we can guarantee during this process the homology stays as torsion free, then we end up with a six manifold X^- that is torsion free and only H^0, H^3 and H^6 are non-vanishing. By the classification theorem of Wall [9], it is diffeomorphic to connected sum of $S^3 \times S^3$. It transforms to S^6 by a sequence of positive conifold transitions (example 6.2).

Because X^+ is simply connected, every class in $H_2(X^+)$ can be represented by S^2. Pick a S^2 that represents a primitive homology class. We can cover X^+ by open subspaces U^+ and V^+ where U^+ is the unit 4-disc bundle over S^2 and V^+ is the complement of a radius $1-\epsilon$ 4-disc bundle over S^2 inside X^+ for $\epsilon > 0$. The intersection $U^+ \cap V^+$ deformation retracts to $S^2 \times S^3$. There is a Mayer-Vietoris sequence of homology

$$0 \longrightarrow H_4(V^+) \longrightarrow H_4(X^+) \longrightarrow$$

$$H_3(U^+ \cap V^+) \cong \mathbb{Z} \xrightarrow{0} (0 = H_3(U^+)) \oplus H_3(V^+) \longrightarrow H_3(X^+) \longrightarrow$$

$$H_2(U^+ \cap V^+) \cong \mathbb{Z} \xrightarrow{(1,1)} (\mathbb{Z} \cong H_2(U^+)) \oplus H_2(V^+) \longrightarrow H_2(X^+) \longrightarrow 0$$

The maps $H_2(U^+ \cap V^+) \to H_2(U^+)$ and $H_2(U^+ \cap V^+) \to H_2(V^+)$ are isomorphisms and the map $H_3(U^+ \cap V^+) \to H_3(V^+)$ is zero because S^2 represents a primitive class in X^+ and S^3 represents zero class. Therefore, V^+ is torsion free with betti numbers

$$b_2(V^+) = b_2(X^+); b_3(V^+) = b_3(X^+); b_4(V^+) = b_4(X^+) - 1.$$

On the other hand, X^- can be covered by open subspaces U^- and V^-, where U^- is the unit 3-disc bundle over S^3 and V^- is the complement of a $(1-\epsilon)$ 3-disc bundle inside X^-. The intersection $U^- \cap V^-$ deformation retracts to $S^3 \times S^2$ and V^- is homeomorphic to V^+. There is a Mayer-Vietoris sequence:

$$0 \longrightarrow H_4(V^-) \longrightarrow H_4(X^-) \longrightarrow$$

$$H_3(U^- \cap V^-) \cong \mathbb{Z} \xrightarrow{(1,0)} (\mathbb{Z} = H_3(U^-)) \oplus H_3(V^-) \longrightarrow H_3(X^-) \longrightarrow$$

$$H_2(U^- \cap V^-) \cong \mathbb{Z} \xrightarrow{1} (0 \cong H_2(U^-)) \oplus H_2(V^-) \longrightarrow H_2(X^-) \longrightarrow 0$$

Since V^+ has been proved to be torsion free and V^- is homoemorphic to V^+, X^- is torsion free as well by MV sequence.

We have the betti number relations:

$$b_2(V^-) = b_2(X^-) + 1; b_3(V^-) = b_3(X^-); b_4(V^-) = b_4(X^-).$$

In total, we get

$$b_2(X^-) = b_2(X^+) - 1; b_3(X^-) = b_3(X^+); b_4(X^-) = b_4(X^+) - 1.$$

This implies the vanishing 3-sphere has trivial homology class in X^-. Repeating negative conifold transitions, we eventually get connected sum of $S^3 \times S^3$ by Wall's theorem. Since it never involves positive even multiple of a primitive 3-cycle, it is an admissible CY 3-fold.

If the S^2 represents a trivial homology class then $H_2(V^- \cap U^-) \to H_2(V^-)$ will be zero and the map $H_3(V^- \cap U^-) \to H_3(V^-)$ will be an isomorphism. In this case the betti number relations are:

$$b_2(X^-) = b_2(X^+); b_3(X^-) = b_3(X^+) + 2; b_4(X^-) = b_4(X^+).$$

This happens when we do negative conifold transition from S^6 to $S^3 \times S^3$. □

Corollary 6.11. *If Y is a simply connected torsion free CY 3-fold then $\tilde{\mathscr{B}}_Y$ is simply connected.*

Proof. The previous theorem shows that such CY can be constructed from connected sum of k copies of $S^3 \times S^3$ by a sequence of positive conifold transitions. By Theorem 6.4 and 6.5, if $\tilde{\mathscr{B}}_{X^-}$ is simply connected so is $\tilde{\mathscr{B}}_{X^+}$. Therefore, it suffices to show it for Y being the k connected sum of $S^3 \times S^3$.

There is a cofibration

$$\bigvee_{2k} S^3 \hookrightarrow Y \longrightarrow S^6.$$

It induces a fibration

$$\tilde{\mathscr{B}}_{S^6} \longrightarrow \tilde{\mathscr{B}}_Y \longrightarrow \prod \tilde{\mathscr{B}}_{S^3}.$$

Because $\tilde{\mathscr{B}}_{S^6}$ and $\tilde{\mathscr{B}}_{S^3}$ are both simply connected, the corollary follows from the long exact sequence of homotopy groups. □

Remark 6.12. Let P_Σ be a simply connected and torsion free toric variety. Any generic complete intersection CY 3-fold in P_Σ is simply connected and torsion free by Lefschetz hyperplane theorem and universal coefficients theorem. For instance, a generic quintic threefold in $\mathbb{P}^4$ is simply connected and torsion free.

Fix positive integers n and m, we can define a substack of the product of moduli stack of vector bundles $Vect_X(n) \times Vect_X(m+n)$ by

$$Pa_{n|m} := \{(E, F) | E \subset F\}$$

where E and F are vector bundles of rank n and $n+m$. By forgetting the holomorphic condition, we may consider the space of equivalence classes of framed flags of principal bundles, denoted by $\tilde{\mathscr{P}}_{X,n|m}$.

Corollary 6.13. *If Y is an admissible CY 3-fold, $H^2(\tilde{\mathscr{P}}_{Y,n|m}, \mathbb{Z})$ is free of even torsion for $n \gg 0$ and $m \gg 0$.*

Proof. There is a weak homotopy equivalence

$$\tilde{\mathscr{P}}_{Y,n|m} \simeq Map^*(Y, BG)$$

for $G = \begin{pmatrix} SL(n,\mathbb{C}) & * \\ 0 & SL(m,\mathbb{C}) \end{pmatrix}$.

The right hand side is homotopic to $Map^*(Y, BSU(n)) \times Map^*(Y, BSU(m))$. The corollary then follows from 6.9. □

7 Orientation data

In this section, we will assume X to be projective.

7.1 Definition of orientation data

We recall the definition of orientation data on a CY 3-category $\mathcal{C}$ following [12]. The definition of Calabi-Yau category can be found in section 3.3 of [12]. Since we will not discuss the general CY categories, the reader could assume $\mathcal{C}$ is the derived category of coherent sheaves of a CY 3-fold. For technical reason, we fix a particular t-structure on $\mathcal{C}$ and all the objects are assumed to belong to the heart of the chosen t-structure. When $\mathcal{C} = \mathrm{D}^b(X)$, the t-structure is chosen to be the standard t-structure of coherent sheaves unless we specify. Denote the moduli stack of objects in the heart by $Ob(\mathcal{C})$. It coincides with $\mathcal{M}$ in the previous notation. By general theory of Artin stacks, one can find a stratification of $Ob(\mathcal{C})$ by locally closed substacks which are of the form $[Y_i/GL(N_i)]$ for $i \in I$. There is a (ind-)constructible determinant line bundle $\mathscr{L}$ defined over $Ob(\mathcal{C})$. Let $E_1 \to E_2 \to E_3$ be a short exact sequence. We have isomorphism of (ind-)constructible line bundles

$$\mathscr{L}_{E_2} \otimes \mathscr{L}_{E_1}^{-1} \otimes \mathscr{L}_{E_3}^{-1} \simeq \mathscr{L}_{E_1 \oplus E_3} \otimes \mathscr{L}_{E_1}^{-1} \otimes \mathscr{L}_{E_3}^{-1} \simeq (sdet(\mathrm{Ext}^\bullet(E_1, E_3)))^{\otimes 2}.$$

Definition 7.1. (Definition 15 of [12]) Orientation data on $\mathcal{C}$ consists of a choice of an (ind-)constructible super line bundle $\sqrt{\mathscr{L}}$ on $Ob(\mathcal{C})$ such that its restriction to each Y_i, $i \in I$ is $GL(N_i)$-equivariant, endowed on each Y_i with $GL(N_i)$-equivariant isomorphisms $(\sqrt{\mathscr{L}})^{\otimes 2} \simeq \mathscr{L}$ and such that for the natural pull-backs to the ind-constructible stack of exact triangles $E_1 \to E_2 \to E_3$ we are given equivariant isomorphisms:

$$\sqrt{\mathscr{L}}_{E_2} \otimes (\sqrt{\mathscr{L}}_{E_1})^{-1} \otimes (\sqrt{\mathscr{L}}_{E_3})^{-1} \simeq sdet(\mathrm{Ext}^\bullet(E_1, E_3))$$

such that the induced equivariant isomorphism

$$\mathscr{L}_{E_2} \otimes \mathscr{L}_{E_1}^{-1} \otimes \mathscr{L}_{E_3}^{-1} \simeq (sdet(\mathrm{Ext}^\bullet(E_1, E_3)))^{\otimes 2}$$

coincides with the one which we have a priori.

The first part of the definition is about the existence of a square root of the determinant bundle. Let $\mathcal{M}^{(2)}$ be the moduli stack of short exact sequences in $Ob(\mathcal{C})$.

$$\begin{tikzcd} \mathcal{M}^{(2)} \arrow[r, "b"] \arrow[d, "(a_1,a_3)"] & \mathcal{M} \\ \mathcal{M} \times \mathcal{M} & \end{tikzcd}$$

The pullback pushforward functor defines the associated product in motivic Hall algebra. The second part of the definition essentially says the square root is compatible under the associated product of motivic Hall algebra (see [12] for definition). We call this condition the *Hall algebra identity*. This is an identity of line bundles on $\mathcal{M}^{(2)}$. To be more precise, it should be written as

$$b^*\sqrt{\mathscr{L}}_{E_2} \otimes a_1^*(\sqrt{\mathscr{L}}_{E_1})^{-1} \otimes a_3^*(\sqrt{\mathscr{L}}_{E_3})^{-1} \simeq sdet(\mathrm{Ext}^\bullet(E_1, E_3)).$$

As we see in Section 2, the determinant line bundle carries holomorphic (algebraic) structure. We consider a variation of Kontsevich-Soibelman's definition.

Definition 7.2. Let $\mathscr{L}$ be the determinant line bundle over $Ob(\mathcal{C})$ with a structure of holomorphic line bundle described in section 2. We have an isomorphism of holormophic line bundles:

$$\mathscr{L}_{E_2} \otimes \mathscr{L}_{E_1}^{-1} \otimes \mathscr{L}_{E_3}^{-1} \simeq (sdet(\mathrm{Ext}^\bullet(E_1, E_3)))^{\otimes 2}.$$

Orientation data on $\mathcal{C}$ consists of a choice of a holomorpihc line bundle $\sqrt{\mathscr{L}}$ on $Ob(\mathcal{C})$ such that $(\sqrt{\mathscr{L}})^{\otimes 2} \simeq \mathscr{L}$ and for the natural pull-backs to the moduli stack of exact triangles $E_1 \to E_2 \to E_3$ we are given isomorphisms of complex $\mathscr{C}^\infty$ line bundles:

$$\sqrt{\mathscr{L}}_{E_2} \otimes (\sqrt{\mathscr{L}}_{E_1})^{-1} \otimes (\sqrt{\mathscr{L}}_{E_3})^{-1} \simeq sdet(\mathrm{Ext}^\bullet(E_1, E_3))$$

such that the induced isomorphism

$$\mathscr{L}_{E_2} \otimes \mathscr{L}_{E_1}^{-1} \otimes \mathscr{L}_{E_3}^{-1} \simeq (sdet(\mathrm{Ext}^\bullet(E_1, E_3)))^{\otimes 2}$$

coincides with the one which we have a priori.

One could consider the stronger version by requiring the Hall algebra identity holds holomorphic. In fact, if a holomorphic line bundle has a $\mathscr{C}^\infty$ (even topological) square root then its square root is also holomorphic. However, in order to show the isomorphism in the Hall algebra identity is an isomorphism of holomorphic line bundles, additional work needs to be done.

7.2 A theorem of Joyce-Song

As we have mentioned in remark 2.3, the analytic definition of determinant line bundle is applicable only to vector bundles. We need to prove that $\mathscr{L}$ over

moduli of vector bundles determines $\mathscr{L}$ over $Ob(\mathcal{C})$. This is a consequence of an important theorem of Joyce and Song. We will review this theorem and several corollaries of it.

Theorem 7.3. *(Theorem 5.3 of [11]) Given a bounded family of coherent sheaves $\mathcal{M}$ over X, there exists an auto-equivalence Φ of $\mathrm{D}^b(X)$ such that Φ induces an open immersion of Artin stacks from $\mathcal{M}$ to $Vect_X$.*

Here is the sketch of the proof. Let Φ_N be the functor of spherical twist at the line bundle $\mathcal{O}_X(-N)$. The first step is to prove Φ_N pulls back universal family. Given a bounded family $\mathcal{M}$, one can choose N big enough such that the image of Φ_N are sheaves. Then one need to show that the tor dimension of the image can be reduced by choosing bigger N.

Suppose we have such an open immersion $\Phi_N : \mathcal{M} \to Vect_X$, the following corollary is an immediate consequence of the fact that Φ_N pulls back universal family.

Corollary 7.4. *There is an isomorphism of line bundles $\Phi_N^* \mathscr{L}_{Vect_X} \cong \mathscr{L}_{\mathcal{M}}$.*

So we can embed any bounded family of coherent sheaves into moduli of vector bundles and restrict the determinant line bundle. Moreover, the restriction is independent of different choices of embedding. Since the rank of the image of Φ_N increases when we choose bigger N, it suffices to consider moduli space of vector bundles of large enough rank. We have already used this fact implicitly in Section 4. Recall when we prove theorem 4.2 we only consider $\tilde{\mathscr{B}}_{SU(n)}$ for large n.

The following statement follows from Theorem 7.3 as well.

Corollary 7.5. *Let $E_1 \to E_2 \to E_3$ be a bounded family of short exact sequences of sheaves on X. There exists $N \gg 0$ such that Φ_N induces an open immersion from this family into the moduli stack of short exact sequences of vector bundles.*

7.3 Existence of orientation data

Theorem 7.6. *If X is an admissible CY 3-fold, then the orientation data on $\mathrm{D}^b(X)$ exists. If X is simply connected and torsion free then the orientation data exists uniquely.*

Proof. By theorem 7.3 and corollary 7.4, the determinant line bundle on $\mathcal{M}_X$ is the pull back of the determinant line bundle on $Vect_X(n)$ for $n \gg 0$. Moreover, $Vect_X(n)$ is embedded inside $\mathscr{B} := \mathscr{B}_{X,SU(n)}$ and the determinant line bundle extends to $\mathscr{B}$.

By Theorem 5.1, the rational $c_1(\mathscr{L})$ is divisible by two. If $\mathscr{B}$ has no even torsion in second cohomology group then the determinant line bundle $\mathscr{L}$ admit a square root. If we consider $\tilde{\mathscr{B}}$ instead of $\mathscr{B}$ then vanishing of 2-torsion follows from corollary 6.9.

Recall that $\mathscr{B}$ is a quotient of $\tilde{\mathscr{B}}$ by $G = SU(n)$. Since the homotopy fiber space $\tilde{\mathscr{B}} \times_G EG$ is a free quotient of $\tilde{\mathscr{B}} \times EG$, the fiberation

$$G \to \tilde{\mathscr{B}} \times EG \to \tilde{\mathscr{B}} \times_G EG$$

induces a spectral sequence of cohomology. Because G is 2-connected, the first and the second cohomologies of $\tilde{\mathscr{B}}$ and $\tilde{\mathscr{B}} \times_G EG$ coincide.

Now we check the Hall algebra identity. By Corollary 7.5, it suffices to check for short exact sequences of vector bundles. Suppose the Hall algebra identity doesn't hold, then

$$\sqrt{\mathscr{L}}_{E_2} \otimes (\sqrt{\mathscr{L}}_{E_1})^{-1} \otimes (\sqrt{\mathscr{L}}_{E_3})^{-1} \otimes sdet(\mathrm{Ext}^\bullet(E_3, E_1))$$

defines a non-trivial 2-torsion line bundle on $\tilde{\mathscr{P}}_{Y,n|m}$. Because $\mathscr{C}^\infty$ 2-torsion line bundles are classified by 2-torsion part of $H^2(\tilde{\mathscr{P}}_{Y,n|m}, \mathbb{Z})$, it must be trivial by corollary 6.13.

Given the existence, the spin structures are classified by $H^1(\tilde{\mathscr{B}}, \mathbb{Z}/2\mathbb{Z})$. For Y simply connected and torsion free, $\tilde{\mathscr{B}}_Y$ is simply connected (corollary 6.11). $\square$

Remark 7.7. In the triangulated category $\mathrm{D}^b(X)$, there are many interesting t-structures besides $coh(X)$. For example, the t-structures of perverse coherent sheaves. The objects in the heart of such a t-structure are complexes of coherent sheaves satisfying appropriate conditions. One can also ask the existence of orientation data in the heart of such t-structure. To extend our proof to these cases, one need to do gauge theory on complex of vector bundles. Some fundamental problems are still open in this field.

However, if the new t-structure can be obtained from $coh(X)$ by tilting (in the sense of [10]) then an orientation data on the new heart can be constructed using Hall algebra identity.

8 Orientation data and volume form on Lagrangian distributaion

In this section, we give a geometric interpretation of orientation data by examples. These examples are related to volume forms on Lagrangian sub bundles in obstruction theory of moduli space.

8.1 Obstruction theory and Lagrangian distribution

We first review some aspects of obstruction theory of moduli space of sheaves on CY 3-fold. Much of the material is excellently present elsewhere, e.g. [2].

For simplicity, we restrict to moduli space of simple vector bundles. It is a $\mathbb{C}^*$ gerbe over a scheme (DM stack). By requiring the bundle in the moduli space to have trivial determinant, it can be reduced to a scheme (DM stack). We denote it by $\mathcal{M}^{si}$.

Definition 8.1. Let Y denote a scheme and L_Y be its cotangent complex. A *perfect obstruction theory* for Y is a morphism $\phi : E \to L_Y$ in derived category such that:

(1) $E \in \mathrm{D}(\mathcal{O}_Y)$ is perfect, of amplitude $[-1, 0]$,

(2) ϕ induces an isomorphism on H^0 and an epimorphism on H^{-1}.

A perfect obstruction theory $E \to L_Y$ is called *symmetric* if there is an isomorphism $\theta : E \to E^\vee[1]$ in $\mathrm{D}(\mathcal{O}_Y)$ such that $\theta^\vee[1] = \theta$.

Let X be a smooth projective CY threefold. The moduli space $\mathcal{M}^{si}$ carries a symmetric obstruction theory. Let $\mathcal{E}$ be the universal bundle and $\mathcal{F}$ be the shifted cone of the trace map:

$$\begin{tikzcd} & \mathcal{O} \arrow[dl, "{[1]}"'] & \\ \mathcal{F} \arrow[rr] & & \mathbf{R}\underline{\mathrm{Hom}}(\mathcal{E},\mathcal{E}) \arrow[ul, "tr"'] \end{tikzcd}$$

Denote the projection from $\mathcal{M}^{si} \times X$ to $\mathcal{M}^{si}$ by π. Then $R\pi_*\mathcal{F}[2]$ is a symmetric obstruction theory for $\mathcal{M}^{si}$. Let $[E]$ be a point on $\mathcal{M}^{si}$ represented by a vector bundle E. The fiber of $H^{-1}(R\pi_*\mathcal{F}[2])$ and $H^0(R\pi_*\mathcal{F}[2])$ at E is a graded vector space of trace free extensions. It is not a locally free sheaf over $\mathcal{M}^{si}$ since the dimensions of the extension groups could jump.

Locally, we can add positive eigenspaces of the Laplacian to make it a perfect complex of amplitude $[-1,0]$. Let E be a holomorphic vector bundle, identified with the underlying complex vector bundle together with an integrable $\bar\partial$ connection A. Pick l to be a positive real number that is smaller than λ_1, the first positive eigenvalue of Δ_A. Denote $H^i_{<l}$ for the direct sum of eigenspaces with eigenvalue $\lambda < l$. There exists an open neighborhood U_A over which $H^i_{<l}$ forms a holomorphic vector bundle. Consider the complex of vector bundles

$$H^1_{<l} \xrightarrow{\nabla_A} H^2_{<l}$$

over U_A. It is quasi-isomorphic to $R\pi_*\mathcal{F}[2]|_{U_A}$. Moreover, this complex $H^\bullet_{<l}$ carries a structure of differential graded Lie algebra over the ring of functions on U_A. It is the local finite dimensional model of the DGLA $L = \mathscr{A}^{0,\bullet}(adE)$. However, $H^\bullet_{<l}$ is not quasi-isomorphic to $L|_{U_A}$ as DGLAs. This is because the products of two harmonic forms in general can have nonzero components of arbitrary high eigenvalues.

Proposition 8.2. *There exists L_∞ products $\mu_k : \bigwedge^k H^\bullet_{<l} \to H^\bullet_{<l}[2-k]$ with $\mu_1 = \nabla_A$ such that $(H^\bullet_{<l}, \mu_k)$ is quasi-isomorphic to L as L_∞ algebras over ring of functions on U_A.*

Proof. This is a direct consequence of the decomposition theorem of DGLA, which says a DGLA is quasi-isomorphic to its cohomology with L_∞ structure. We refer to [17] for the proof of the decomposition theorem and the definition of L_∞ algebra.

Apply the decomposition theorem to $L|_{U_A}$. Its cohomology $H^\bullet(L)$ can be identified with $\mathrm{Ext}^\bullet(E,E)$ fiberwise. $H^\bullet_{<l}$ decomposes into direct sum of harmonic part, which can be identified with $H^\bullet(L)$, and positive eigenspace. Since

the second part is acyclic with respect to $\mu_1 = \nabla_A$, the inclusion is a quasi-isomorphism of L_∞ algebras. Now we compose the two quasi-isomorphisms to obtain the quasi-isomorphism from $H^\bullet_{<l}$ to L. $\square$

The Calabi-Yau condition gives a bilinear form κ on L

$$(A, B) \mapsto \int_X Tr(A \wedge B) \wedge \Omega^{3,0}$$

for $A, B \in L$, where $\Omega^{3,0}$ is a holomorphic volume form. κ induces the Serre pairing on $\mathrm{Ext}^\bullet(E, E)$. Under this pairing, $H^1_{<l}$ is dual to $H^2_{<l}$. A more invariant way is to say that $H^\bullet_{<l}$ is a graded vector bundle over U_A with an odd symplectic form κ. The sub bundle $H^1_{<l}$ is Lagrangian. A (holomorphic) volume form on $H^1_{<l}$ is a nowhere vanishing section of $\bigwedge^{top} H^2_{<l}$.

If the moduli space $\mathcal{M}^{si}$ is smooth then $H^1_{<l}$ is a vector bundle globally. It is Lagrangian in $H^\bullet_{<l}$. The determinant bundle of $H^2_{<l}$ is a square root of the determinant line bundle $\mathscr{L}$. Therefore, the square root of $\mathscr{L}$ measures how the volume form on $H^1_{<l}$ is "twisted". In the general case, we cover $\mathcal{M}^{si}$ by charts. Over each chart, there is a vector bundle $H^\bullet_{<l}$ that contains a Lagrangian sub bundle $H^1_{<l}$. Let's call such a collection of $H^1_{<l}$ together with the differential ∇_A a *Lagrangian distribution*.

By a theorem of Joyce and Song (Theorem 5.5 [11]), moduli space of sheaves on a compact CY 3-fold can be locally embedded into $T^*\mathrm{Ext}^1(E, E)$ near a point E. There exists a holomorphic function f on an open neighborhood of E contained in $\mathrm{Ext}^1(E, E)$ such that the local chart of the moduli space is equal to $Crit(f)$. This implies the moduli space is locally intersection of two Lagrangian subvarieties in $T^*\mathrm{Ext}^1(E, E)$, zero section and graph of df. The subspace $Ker\ \nabla_A \subset H^1_{<l}$ can be identified with the tangent space of the zero section.

Now we gave two examples of Lagrangian distributions that come from actual Lagrangian subvarieties. Both are related to degeneration of CY 3-folds.

8.2 local CY 3-folds

Let S be a smooth projective surface and K_S be the total space of its canonical bundle. Consider the subcategory D_ω of $\mathrm{D}^b(K_S)$ consisting of complexes of sheaves support on the zero section. This is a CY 3-category in the sense of [12]. Denote the projection from K_S to S by π and the inclusion of zero section by i. Any sheaf in D_ω can be written as a consequent extension of sheaves of the form i_*E for $E \in coh(S)$. Therefore, to understand the deformation theory of sheaves in D_ω it suffices to understand that of i_*E.

Proposition 8.3. *There is an exact triangle*

$$\begin{tikzcd} & \mathbf{R}\mathrm{Hom}_S(E,E)^\vee[-3] \arrow[dr] & \\ \mathbf{R}\mathrm{Hom}_S(E,E) \arrow[ur, "{[1]}"] & & \mathbf{R}\mathrm{Hom}(i_*E, i_*E) \arrow[ll] \end{tikzcd}$$

Proof. Denote the total space of K_S by X. There is a short exact sequence:

$$0 \longrightarrow \mathcal{O}_X(K_S^{-1}) \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_S \longrightarrow 0.$$

Given a coherent sheaf E on S, let $P^\bullet$ be a locally free resolution of E. By tensor the above short exact sequence with π^*E, we obtain an exact triangle:

$$\begin{tikzcd} \pi^*P^\bullet(K_S^{-1}) \arrow[rr] & & \pi^*P^\bullet \arrow[dl] \\ & i_*E \arrow[ul, "{[1]}"] & \end{tikzcd} .$$

Apply the functor $\mathbf{R}\mathrm{Hom}(-, i_*E)$ and Serre duality, we get the desired exact triangle. ☐

A family version of the above proposition can be proved similarly when we replace E by the universal sheaf $\mathcal{E}$.

The symplectic form κ is nothing but the natural pairing between $\mathbf{R}\mathrm{Hom}_S(E,E)$ and $\mathbf{R}\mathrm{Hom}_S(E,E)^\vee$. It is odd because of the grading shift $[-3]$. The sub complex $\mathbf{R}\underline{\mathrm{Hom}}(\mathcal{E},\mathcal{E})$ is a Lagrangian distribution. This means deforming sheaf inside $S \subset K_S$ is a Lagrangian condition.

One can easily check the determinant line bundle of the complex $\mathbf{R}\underline{\mathrm{Hom}}_S(\mathcal{E},\mathcal{E})$ satisfies the Hall algebra identity, i.e. it is an orientation data on D_ω. Volume form of this Lagrangian distribution can be interpreted as local sections of canonical bundle of the moduli space of sheaves on S. On the other hand, orientation data measures the twist of volume form on the Lagrangian distribution.

8.3 Degeneration of quintic threefold

The following example is took from [18]. Let X_0 be a degeneration of quintic threefold into union of two Fano hypersurfaces Y_1 and Y_2 of degree two and three. Let S be the intersection of Y_1 and Y_2. We assume that Y_1, Y_2 and S are all smooth. The pushout diagram induces a commutative diagram of moduli stacks of sheaves

$$\begin{tikzcd} S \arrow[r, "i_1"] \arrow[d, "i_2"] & Y_1 \arrow[d, "j_1"] \\ Y_2 \arrow[r, "j_2"] & X_0 = Y_1 \cup_S Y_2 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}_{X_0} \arrow[r, "f_1"] \arrow[d, "f_2"] & \mathcal{M}_{Y_1} \arrow[d, "g_1"] \\ \mathcal{M}_{Y_2} \arrow[r, "g_2"] & \mathcal{M}_S \end{tikzcd}$$

We assume that objects in $\mathcal{M}_{X_0}$ are perfect complexes. This guarantees determinant line bundle to be well defined. Let E be a sheaf on X_0. The exact sequence

$$0 \longrightarrow \mathcal{O}_{X_0} \longrightarrow \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

induces an exact triangle of obstruction theories

$$\begin{tikzcd}
& \mathbf{R}\mathrm{Hom}_{X_0}(E,E) \arrow[dr] & \\
\mathbf{R}\mathrm{Hom}(E|_S,E|_S) \arrow[ur, "{[1]}"] & & \mathbf{R}\mathrm{Hom}(E|_{Y_1},E|_{Y_1}) \oplus \mathbf{R}\mathrm{Hom}(E|_{Y_2},E|_{Y_2}). \arrow[ll, "{(\theta_1,\theta_2)}"']
\end{tikzcd}$$

Proposition 8.4. *There is an exact triangle of obstruction theories*

$$\begin{tikzcd}
& \mathbf{R}\mathrm{Hom}(E|_{Y_1},E|_{Y_1})^\vee[-3] \arrow[dr] & \\
\mathbf{R}\mathrm{Hom}(E|_{Y_2},E|_{Y_2}) \arrow[ur, "{[1]}"] & & \mathbf{R}\mathrm{Hom}_{X_0}(E,E) \arrow[ll]
\end{tikzcd}$$

Proof. The short exact sequence

$$0 \longrightarrow K_S \longrightarrow \mathcal{O}_{X_1} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

induces an exact triangle of obstruction theories

$$\begin{tikzcd}
& \mathbf{R}\mathrm{Hom}(E|_{Y_1},E|_{Y_1}) \arrow[dr, "\theta_1"] & \\
\mathbf{R}\mathrm{Hom}(E|_{Y_1},E|_{Y_1})^\vee[-2] \arrow[ur, "{[1]}"] & & \mathbf{R}\mathrm{Hom}(E|_S,E|_S) \arrow[ll]
\end{tikzcd}$$

Apply the octahedron axiom of triangulated category to the inner commutative triangle:

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}(E|_{Y_2},E|_{Y_2}) \arrow[ddrr, dashed] & & \\
\bigoplus_{i=1}^2 \mathbf{R}\mathrm{Hom}(E|_{Y_i},E|_{Y_i})) \arrow[u] \arrow[dr, "{(\theta_1,\theta_2)}"] & & \\
\mathbf{R}\mathrm{Hom}(E|_{Y_1},E|_{Y_1}) \arrow[u, hook] \arrow[r, "\theta_1"] & \mathbf{R}\mathrm{Hom}(E|_S,E|_S) \arrow[r] \arrow[dr] & \mathbf{R}\mathrm{Hom}(E|_{Y_1},E|_{Y_1})^\vee[-2] \arrow[d, dashed, "{[1]}"] \\
& & \mathbf{R}\mathrm{Hom}_{X_0}(E,E)[1].
\end{tikzcd}$$

The octahedron axiom implies the dash arrows form an exact triangle. $\square$

The obstruction theories $\mathbf{R}\mathrm{Hom}(E|_{Y_1},E|_{Y_1})$ and $\mathbf{R}\mathrm{Hom}(E|_{Y_2},E|_{Y_2})$ form Lagrangian distributions inside $\mathbf{R}\mathrm{Hom}_{X_0}(E,E)$. Denote their determinant line bundles by $\mathscr{L}_{Y_1}$ and $\mathscr{L}_{Y_2}$ respectively. The determinant line bundle $\mathscr{L}_{X_0}$ of X_0 satisfies

$$\mathscr{L}_{X_0} \cong \mathscr{L}_{Y_1}|_{\mathcal{M}_{X_0}} \otimes \mathscr{L}_{Y_2}|_{\mathcal{M}_{X_0}}.$$

Remark 8.5. When both $\mathscr{L}_{Y_1}$ and $\mathscr{L}_{Y_2}$ admit square roots, $\mathscr{L}_{X_0}$ has a square root as well. We might compare such condition with a condition in Floer theory that requires the Lagrangian submanifolds L_1 and L_2 that intersect are spin Lagrangian submanifolds. In Floer theory, this condition gives an orientation on the moduli space of holomorphic discs.

Remark 8.6. In a beautiful paper [14], Pantev, Toen, Vaquie and Vezzosi gave a definition of derived symplectic manifold and derived Lagrangian submanifold. We believe the work of PTVV provides the correct framework to discuss the categorified DT theory. In the derived world, the smoothness or transversality assumptions could be removed. In that sense, the previous two examples shows that the orientation data comes from volume form of certain derived Lagrangian submanifold inside the moduli space equipped with a derived symplectic structure.

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